The Geometry of the Gromoll Filtration

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1 The Filtration

Recall that we are interested in the group of homotopy spheres for many reasons. We denote this group Θ_n. Recall that there is an isomorphism of groups (see my notes)

$$\pi_0 \mathrm{Diff}_\partial(D^n) \cong \Theta_{n+1}$$

The right hand side admits a natural filtration. First we define

$$\gamma^{n+1}_{(n-i)} \;\; := \;\; \pi_i \mathrm{Diff}_\partial(D^{n-i})$$

the superscript denotes the dimension of the homotopy spheres that we are defining a subgroup of and the subscript denotes the dimension of the disc that we are looking at, or whats the same the dimension of the sphere that we are pulling back to, more below.

Now it is clear that (**tensor-hom adjunction**)

$$\gamma^{n+1}_{(n-i)} = [S^i, \mathrm{Diff}_\partial(D^{n-i})] \subseteq [S^i, \mathrm{Hom}(D^{n-i}, D^{n-i})] \cong [S^i \times D^{n-i}, D^{n-i}]$$

If we had a map $D^i \to \mathrm{Hom}(D^{n-i}, D^{n-i})$ that was the identity on the boundary of D^i, **that is the constant map that hits the identity (the basepoint) in** $\mathrm{Hom}(D^{n-i}, D^{n-i})$, then we can glue it up to a map $S^i \to \mathrm{Hom}(D^{n-i}, D^{n-i})$ (**universal property of quotient**). It is clear that all maps (preserving the base point) will be of this form. Therefore we can represent an element $\alpha \in \gamma^{n+1}_{(n-i)}$ by a map

$$D^i \times D^{n-i} \to D^{n-i}$$

where the condition that it is the identity on the boundary of D^i can be translated into saying that on the boundary of $D^i \times D^{n-i}$ it is the projection onto the second factor $\alpha(x, y) = y$, that is it is constant on the boundary of the first variable and the thing that it is constant on is the identity map in the second variable.

Hence we have that

$$D^i \times D^{n-i} \cong D^{i-1} \times D^1 \times D^{n-i} \cong D^{i-1} \times D^{n-i+1}$$

while at the same time

$$D^{n-i} \xrightarrow{\iota} D^{n-i} \times D^1 \cong D^{n-i+1}$$

therefore we can send

$$\mathrm{Hom}\left(D^i \times D^{n-i}, D^{n-i}\right) \to \mathrm{Hom}\left(D^{i-1} \times D^{n-i+1}, D^{n-i}\right) \xrightarrow{\iota\circ -} \mathrm{Hom}\left(D^{i-1} \times D^{n-i+1}, D^{n-i+1}\right)$$

which may be called an "assembly" map. We are just rearranging the domain and then including the codomain into a bigger space. Through this proceedure then we have provided a map

$$\lambda_i : \gamma^{n+1}_{(n-i)} \to \gamma^{n+1}_{(n-i+1)}$$

which can be iterated to create a sequence of groups

$$\begin{array}{ccccccccccccccc} \gamma^{n+1}_{(0)} & \to & \gamma^{n+1}_{(1)} & \to & \gamma^{n+1}_{(2)} & \to & \gamma^{n+1}_{(3)} & \to & \gamma^{n+1}_{(4)} & \to \cdots \to & \gamma^{n+1}_{(n-i)} & \to & \gamma^{n+1}_{(n-i+1)} & \to \cdots \to & \gamma^{n+1}_{(n)} = \pi_0 \mathrm{Diff}_\partial(D^n) \\ \| & & \| & & \| & & \| & & & & & & & & \\ 0 & & 0 & & 0 & & 0 & & & & & & & & \end{array}$$

This defines a filtration on $\pi_0 \mathrm{Diff}_\partial(D^n)$ by looking at the images of these maps, these subgroups we denote

$$\Gamma^{n+1}_{(n-i)} \;\; := \;\; \lambda_{n-1} \circ \cdots \circ \lambda_i\big(\gamma^{n+1}_{(n-i)}\big) \subseteq \pi_0 \mathrm{Diff}_\partial(D^n)$$

which gives us the so called Gromoll filtration

$$\Gamma^{n+1}_{(0)} \subseteq \cdots \subseteq \Gamma^{n+1}_{(n-1)} \subseteq \Gamma^{n+1}_{(n)} = \Theta_{n+1}$$

Note that this also produces a filtration of the other groups $\pi_i \mathrm{Diff}_\partial(D^{n-i})$ which by varying the n value produces a filtration of all the homotopy groups of $\mathrm{Diff}_\partial(D^k)$, for an arbitrary k.

Finally we define the **disc of origin**: If $\Sigma \in \Theta_{n+1}$ then its disc of origin is the minimal d such that $\Sigma \in \Gamma^{n+1}_{(d)}$. We denote this $\mathscr{D}(\Sigma) = d$.

Lemma. *For $\Sigma \in \Theta_{n+1}$ we have that*

$$\mathscr{D}(\Sigma) \leq n-1$$

Proof. $\pi_0 \mathscr{C}(D^n) = 0$ and putting this in our LES in homotopy groups for the fibration

$$\mathrm{Diff}_\partial(D^{m+1}) \to \mathscr{C}(D^m) \to \mathrm{Diff}_\partial(D^m)$$

implies a surjection between $\pi_1 \mathrm{Diff}_\partial(D^{n-1}) \to \pi_0 \mathrm{Diff}_\partial(D^n)$.

□

Another way of saying this is that $\Gamma^{n+1}_{(n)} = \Gamma^{n+1}_{(n-1)}$.

Remark. Here are some intuitive remarks that may or may not be true, Im not quite sure, on what pulling back in this filtration means. An isotopy is a path in $\mathrm{Diff}_\partial(M)$, we have seen in our Cerf notes that this is the same as a diffeomorphism of $M \times I$ that commutes with the projection onto the I factor. Elements of $\pi_1 \mathrm{Diff}_\partial(M)$ are in particular paths in $\mathrm{Diff}_\partial(M)$, that start and end at the base point. Thus an element of $\pi_1 \mathrm{Diff}_\partial(M)$ is in particular an isotopy. Now conflating $I^n \cong D^n$ we can see that moving around the I factors in the $M \times I$ will allow us to say more. Indeed when we pull back on the disc it is like going from an isotopy of $I^{n-1} \times I$, that is a diffeomorphism that commutes with the projection to the second factor, to a diffeomorphism of $I^{n-2} \times I^2$ the commutes with the projection on to the second variable. This is something like the diffeomorphism begin "flat" or the identity in more directions.

Remark. The injectivity of this map $\pi_1 \mathrm{Diff}_\partial(D^{n-1}) \to \pi_0 \mathrm{Diff}_\partial(D^n)$ is of interest and [Wan24] has provided some cases where it is injective.

2 Exotic Blobs

Let (A, B) be a pair of closed smooth manifolds, $B \subseteq A$. Then define

$$\mathscr{S}(A, B) \quad := \quad \left\{f : M \xrightarrow{\simeq} A;\ \text{rel } B\right\} / \sim$$

that is closed smooth manifolds that are homotopy equivalent to A relative to B, up to some equivalence. The equivalence is that $f_1 : M_1 \sim f_2 : M_2$ if and only if there is a homtopy commutative diagram,

$$\begin{tikzcd} M_1 \arrow[d] \arrow[dr, "f_1"] & \\ M_2 \arrow[r, "f_2"'] & A \end{tikzcd}$$

such that the map $M_1 \to M_2$ is a diffeomorphism, and when we restrict the diagram to the boundary it commutes up to isotopy. This is written down in [LM24, Def 11.5]. The relevant space for us is

$$\mathscr{S}_\partial(D^{n+1}) \quad := \quad \mathscr{S}(D^{n+1}, \partial D^{n+1})$$

because

Lemma ([LM24], Lem 12.6)**.**

$$\Theta_{n+1} \cong \mathscr{S}(S^{n+1}, \emptyset) \cong \mathscr{S}_\partial(D^{n+1})$$

Proof. The second iso is just the universal property of the quotient.

The first bijection seems tautological, however there is a subtelty, in the KM definiton of Θ we need to account for orientations, whilst $\mathscr{S}$ does not. By the generalised Poincare conjecture a homotopy equivalence is a homeomorphism and moreover up to homotopy between a homotopy sphere and the standard sphere there is exactly two such homeomorphisms (given by the maps of degree ± 1). Therefore given an element in the structure set we can define an orientation on the domain by pulling back the standard orientation on the sphere. This makes the two maps into two different oriented structures and therefore preserves the bijection.

□

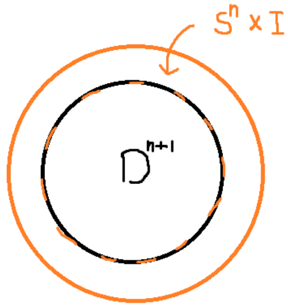
It is clear that one way to exhibit $\mathscr{S}_\partial(D^{n+1})$ is as

$$\Big\{[W] : W \simeq *, \partial W = S^n\Big\}$$

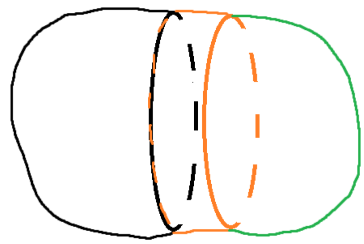
that is closed smooth manifolds that are contractable and have boundary that is the standard smooth sphere identified up to boundary preserving (the identity) diffeomorphism. This is the set of (non-standard name) "exotic blobs". These blobs can all be constructed as gluing collars onto the boundary of standard discs (the proof is the same as the proof for exotic spheres)

$$W_{e(f)} \quad := \quad D^{n+1} \cup_f (I \times S^n), \quad f : S^n \xrightarrow{\sim} S^n$$

It is also clear that if we glue onto the boundary of $W_{e(f)}$ along the identity map we get the exotic sphere that we would have obtained by gluing D^{n+1} to itself along f on its boundary.



Exotic Blob



Exotic Sphere

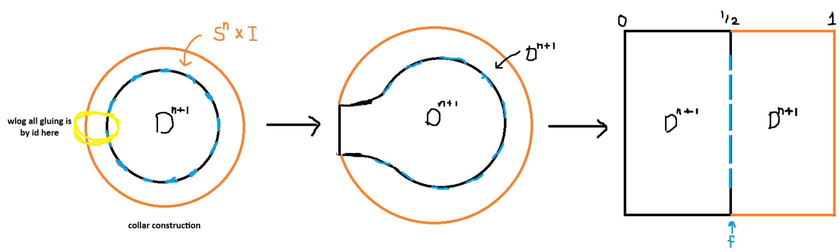
Lemma. *Let $f \in \mathrm{Diff}_\partial(D^n)$ then we may construct*

$$W_f \quad := \quad [0,1/2] \times D^n \cup_f [1/2,1] \times D^n$$

by identifying $D^n \times \{1/2\}$ along f. We claim that

$$[W_f] = [W_{e(f)}] \in \mathscr{S}_\partial(D^{n+1})$$

Proof. We have intuitively drawn the diffeomorphism below, anything more would be too painfully explicit. Because really the two manifolds we have defined $W_{e(f)}, W_f$ are not naturally given with homotopies to the standard disc, we are free to choose them such that they commute trivially in the diagram, namely pick one for the first and then pull it back along the diffeomorphism to give the second. By doing this the diagram commutes strictly and hence also on its boundary. This is sufficient for our purposes. □



The key observation is that if we take a homotopy sphere and cut out a disc, say to for the connected sum then what we are left with is exactly $W_{e(f)}$.

Remark. Note that all discs are diffeomorphic, the point here is whether or not that diffeomorphism is the identity on the boundary.

3 The Geometry of the Filtration

Here are some results and a conjecture that give a geometric description of what this filtration means for the homotopy spheres.

Lemma. *If $\Sigma \in \Gamma^{n+1}_{(n-i)}$ then there exists a fibration*

$$S^{n-i} \to \Sigma \sharp (S^{n-i} \times S^{i+1}) \to S^{i+1}.$$

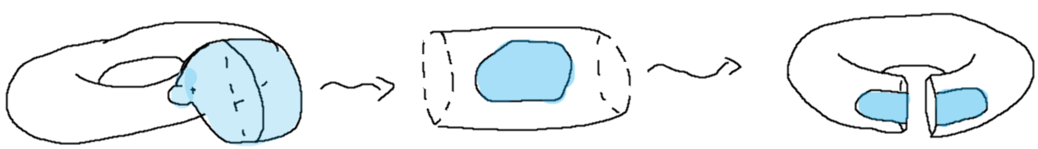
Proof. Consider an element $\xi \in \pi_i \mathrm{Diff}_\partial(D^{n-i})$ which corresponds under the Gromoll assebly maps to $\Sigma \in \Gamma^{n+1}_{(n-i)}$. Then just as in the proof for the bijection of $\pi_0 \mathrm{Diff}_\partial(D^n) \cong \pi_0 \mathrm{Diff}(S^n)$ we can extend pointwise the image of the map $[S^i, \mathrm{Diff}_\partial(D^{n-i})]$ by the identity to get a map of the sphere, thus defining a map

$$[S^i, \mathrm{Diff}_\partial(D^{n-i})] \xrightarrow{[S^i, \text{extend by the identity}]} [S^i, \mathrm{Diff}(S^{n-i})]$$

Let $\sigma \in \pi_i \mathrm{Diff}(S^{n-i})$ correspond to ξ under this map. Then we can form a bundle from σ by clutching. First we want the domain of σ to represent the boundary of the manifold that we will glue, so here we will clearly choose D^{i+1}, and then because we are getting diffeomorphisms of S^{n-i} this should be the fibers of the bundle. Thus we get a priori a bundle

$$S^{n-i} \to E_\sigma \to D^{i+1} \cup D^{i+1} = S^{i+1}.$$

Now our goal is to exibit this total space as diffeomorphic to a connected sum. First we will examine the connected sum pictorially in low dimensions for some intuition. These statements might not be strictly true. Consider $i = 0, n = 1$. Then the bundle is given by two maps from the point to $\mathrm{Diff}_\partial(D^1)$, but because it is pointed maps one is the identity. Thus we can consider the total space as a cylinder with the gluing instructions telling us how to stick the two ends together (we have circles at every point on a one dimensional line, this is a cylinder, we have two copies but one of the ends is glued by the identity). This agrees with the idea that the total space should be $\S^1 \times S^1$ with some homotopy sphere connected. Now the way to connect the homotopy sphere, up to diffeomorphisms is illustrated in this example. Fist we connect sum, then we flatten the sphere down a bit, then we identify it with a peice coming from the gluing up of the bundle



More generally we would like to think of some pair of D^{n+1}'s (the blue bubbles in the final image) existing in the total space of the bundle, *before gluing up*, that then have parts of their boundary identified along the gluing map, which is precisely gluing together the homotopy sphere Σ. To make this more precise consider that E_σ must be given by

$$E_\sigma = \left(S^{n-i} \times D^{i+1}\right) \cup_\sigma \left(S^{n-i} \times D^{i+1}\right)$$

and the gluing is an identification (where in the second component it is the identity, this is similar to the situtaion in our Cerf notes, a path is the same as a map into the product *on both sides*, where the product is the identity map)

$$\sigma : \left(S^{n-i} \times \partial D^{i+1}\right) \to \left(S^{n-i} \times \partial D^{i+1}\right)$$

But σ comes from a map on the disc that is the identity near the boundary and so we can think of it on the boundary as

$$\xi : \left(D^{n-i} \times \partial D^{i+1}\right) \to \left(D^{n-i} \times \partial D^{i+1}\right)$$

where it is the identity near the boundary of D^{n-i} (the boundary is the parameter and the D^{n-i} is the fiber). We can also assume that the parameter is the identity on "the zero section", that is because the homotopy class is pointed maps we know that at the base point they take the identity

and moreover we can assume this in a nieghbournhood of the identity. Thus we can again assume that σ is of the form
$$\xi' : \left(D^{n-i} \times D^i\right) \to \left(D^{n-i} \times D^i\right)$$
such that it is the identity on the boundary of both the D^m components. Now in a collar neighbourhood of this boundary we are gluing

$$D^1 \times D^{n-i} \times D^i \cong D^{n+1}$$

to itself along the $D^{n-i} \times D^i \cong D^n$ component. This D^n is just the S^n that we glue up to homotopy sphere that has been punctured (or extended by the identity).

□

Proof. Using the language of the previous section it is clear that our construction above produces the disc W_f in the total space of the bundle. Then this is the same as the collar construction which is the same as the exotic sphere with a standard disc cut out.

Remark. Again we can see here that pulling back in the filtration means that you can be attained from some $W_{e(f)}$, where this twisted disc has some extra structure. Namely it is given by gluing two D^{n+1}'s but the fibers are twisting, not the base space D^i.

4 A Converse

Given an $n+1$ manifold its **Inertia group** is defined as

$$I(M) = \{\Sigma \in \Theta_{n+1} : M \sharp \Sigma \cong M\}$$

those are the homotopy spheres that sort of act trivially on the manifold. This could be frased as the stabiliser of the element M (say as an element of the set of smooth $n+1$ manifolds) under the action of the group of homotopy spheres given by connect sum.

Lemma ([Sch], [Sap69], [Lev70])**.** *For all p, q the inertia group of $S^p \times S^q$ is trivial.*

This implies that the fibration above is *non-trivial* for all non-standard homotopy spheres (not the standard smooth structure) and that moreover the diffeomorphism type of the homotopy sphere is determined by the diffeomorphism type of this bundle, this is because if

$$\Sigma_1 \sharp \left(S^{n-i} \times S^{i+1}\right) \cong \Sigma_2 \sharp \left(S^{n-i} \times S^{i+1}\right)$$

then using the group structure on the group of homotopy spheres then we get

$$\Sigma_2^{-1} \sharp \Sigma_1 \sharp \left(S^{n-i} \times S^{i+1}\right) \cong \left(S^{n-i} \times S^{i+1}\right)$$

but because the intertial group is trivial this means that

$$\Sigma_2^{-1} \sharp \Sigma_1 \cong 1$$

or in other words the two homotopy spheres are diffeomorphic. Note that we can perform these connect sums away from any exotic structure so this is all fine (the group structure on the group of homotopy spheres can sort of take place independent of the fact that there is a big product of spheres sitting there).

Conjecture 1. *If there exists a fibration*

$$S^{n-i} \to \Sigma \sharp (S^{n-i} \times S^{i+1}) \to S^{i+1}$$

and some obstructions on the normal bundle Schang-Levine-Skarba... vanish then $\Sigma \in \Gamma^{n+1}_{(n-i)}$.

We now have this in a stable range. Lets get some lemmas down first.

Lemma ([ABK72], Lemma 1.1.5, or here.)**.**

$$\pi_i \mathrm{Diff}(S^n) = \pi_i\big(\, \mathrm{SO}_{n+1} \times \mathrm{Diff}_\partial(D^{n-i})\big)$$

Now we are interested in sphere bundles over a sphere, all bundles over the sphere will come from clutching relevant bundles over the disc. Thus we are dealing with a bundle

$$E_\xi = (D^{i+1} \times S^{n-i}) \cup_\xi (D^{i+1} \times S^{n-i})$$

where $\xi : \partial D^{i+1} \to \mathrm{Diff}(S^{n-i}) \in \pi_i \mathrm{Diff}(S^{n-i})$. We can use the universal property of the product to rewrite ξ as

$$\xi = \sigma \times \gamma \in \pi_i\, \mathrm{SO}_{n-i+1} \oplus \pi_i \mathrm{Diff}_\partial(D^{n-i})$$

Lemma. *If $\sigma \neq 0$ and $i << n$ (we are in the stable range of $\pi_i\, \mathrm{SO}_{n-i+1}$) then TE_ξ is stably non-trivial.*

Consider the vector bundle $E_\sigma \to S^{i+1}$ associated with the clutching function σ. First we claim that TE_ξ is stably isomorphic to $\pi^* E_\sigma$, where $E_\xi \xrightarrow{\pi} S^{i+1}$ is the bundle map.

Next E_σ has an associated sphere bundle given by the unit length vectors in the fibers, $SE_\sigma \xrightarrow{\omega} S^{i+1}$. Then Diarmuid claims (in a paper and to me) that there is a stable isomorophism of bundles

$$T(SE_\sigma) \simeq \omega^* T(S^{i+1}) \oplus \omega^* E_\sigma$$

Now according to [KM63, Thm 3.1] all homotopy spheres have a stably trivial tangent bundle, and so we have stable isomorphism

$$T(SE_\sigma) \simeq \omega^* \underline{\mathbb{R}^m} \oplus \omega^* E_\sigma \simeq \omega^* E_\sigma$$

The situation is summarised in the following commuting diagram

$$\begin{tikzcd}
\pi^* E_\sigma \simeq TE_\xi \arrow[d] & E_\sigma \arrow[l, dashed, "\pi^*"'] \arrow[r, dashed, "\omega^*"] \arrow[d] & T(SE_\sigma) \simeq \omega^* E_\sigma \arrow[d] \\
E_\xi \arrow[r, "\pi"] & S^{i+1} & SE_\sigma \arrow[l, "\omega"'] \arrow[ul, hook]
\end{tikzcd}$$

Now in the case that $\gamma = 0$ then it is clear from the construction of E_ξ and SE_σ as clutched vector bundles that they are the same bundle. Hence we can conclude that

$$\pi^* E_\sigma \simeq TE_\xi \simeq T(SE_\sigma) \simeq \omega^* E_\sigma$$

stably as bundles over $E_\xi = SE_\sigma$. Is one of these things obviously stably non-trivial? I think the point is merely that this is how you might prove the first statement? If we are in the stable range the it is clear that E_σ itself is non-trivial and therefore stably non-trivial, because $\sigma \neq 0 \in \pi_j SO$. So what we need to show is that pulling back an unstable bundle is unstable.

Proof. Consider the vector bundle $E_\sigma \to S^{i+1}$ associated with the clutching function σ. First we claim that TE_ξ is stably isomorphic to $\pi^* E_\sigma$, where $E_\xi \xrightarrow{\pi} S^{i+1}$ is the bundle map.

Looking then at E_σ it is given by an element $\sigma \in \pi_i\, \mathrm{SO}(n-i+1)$, by the hypothesis that we are in the stable range however there is an isomorphism between $\pi_i\, \mathrm{SO}(n-i+1)$ and $\pi_i\, \mathrm{SO}$ and therefore σ is non-zero in $\pi_i\, \mathrm{SO}$, these elements clasify stable bundles and therefore E_σ is stably non-trivial.

Pulling back defines a map from bundles over S^{i+1} to bundles over E_ξ, or in KO theory

$$KO(S^{i+1}) \xrightarrow{K(\pi)} KO(E_\xi)$$

and what remains to see is that this map preserves the non-triviality of E_σ. What this amounts to is requiring this map to be injective, as we are saying that a non-zero element goes to a non-zero element.

Because we are in the stable range we know that the bundle E_σ has a section (the dimension of the fiber is greater than the base), i.e. there exists a map $r : S^{i+1} \to E_\sigma$ such that $\pi \circ r = \mathrm{id}_{S^{i+1}}$. On K theory then we get a left inverse to $K(\pi)$

$$\begin{tikzcd} K(S^{i+1}) \arrow[loop left, "\mathrm{id}=K(\pi\circ r)"] \arrow[r, shift left, "K(\pi)"] & K(E_\xi) \arrow[l, shift left, "K(r)"] \end{tikzcd}$$

Therefore we conclude that the map on K theory is an injection and we get that the stably non-trivial bundle E_σ pulls back to something stably non-trivial.

Crucially this proof relies on being in the stable range. If we were not in the stable range then we would loose two things, the fact that the bundle E_σ is necissarily stably non-trivial and the fact that we have a section. Note that this implies that outside the stable range stable triviallity is not a sufficient invariant, it does not show that the converse of the lemma doesnt hold. Thus it is still possible at this point to hope for an unconditional converse.

Lemma. *$T\big(\Sigma\sharp(S^{n-i} \times S^{i+1})\big)$ is stably trivial when $\Sigma \in \Theta_{n+1}$.*

Proof. According to [GK19, Lem 2.1] for connected sums we have a stable isomorphism

$$T(M\sharp N) \simeq T(M) \oplus T(N)$$

and hence in our case we get that

$$T\big(\Sigma\sharp(S^{n-i} \times S^{i+1})\big) \simeq T(\Sigma) \oplus T(S^{n-i} \times S^{i+1}) \simeq T(\Sigma) \oplus T(S^{n-i}) \oplus T(S^{i+1})$$

which is the direct sum of three stably trivial bundles by [KM63, Thm 3.1] and hence itself stably trivial. $\square$

I think D. was having something in mind about the derivative map in order to show this. Together these two imply the following

Theorem. *If $\Sigma \in \Theta_{n+1}$, $i << n$ (we are in the stable range of $\pi_i \mathrm{SO}_{n-i+1}$) and there is a fibration*

$$S^{n-i} \to \Sigma\sharp(S^{n-i} \times S^{i+1}) \to S^{i+1}$$

then $\Sigma \in \Gamma^{n+1}_{(n-i)}$.

Proof. Denote the total space $E_\xi = \Sigma\sharp(S^{n-i} \times S^{i+1})$ for $\xi = \sigma \times \gamma$ as above. Then by the second lemma E_ξ is stably trivial. Then by the (contrapositive of the) first lemma $\sigma = 0$. Thus we conclude that the bundle is glued together using only the γ diffeomorphism, in exactly the way we discussed in the converse. In particular take γ, map it to a homotopy sphere under the Gromoll map and then applying the proof of the converse and the uniqueness of the diffeomorphism type of the bundle we conclude that E_ξ is diffeomorphic to the bundle constructed in the proof of the converse. $\square$

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